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Stochastic mechanics of a Dirac particle in two spacetime dimensions

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Abstract. We develop the stochastic mechanics of a Dirac particle interacting with an arbitrary external electromagnetic field in two-dimensional Minkowski space. Our construction provides a consistent stochastic interpretation of solutions of the true (real time) Dirac equation and, in particular, gives a stochastic description of the zitterbewegung.

1. Introduction

Under the influence of the classical work of Nelson, stochastic mechanics has been a considerably active subject in recent years. In our opinion, what is at stake in this type of research is to establish whether any physical system which can be described within quantum mechanics admits a classical probabilistic description.

Nelson considered the usual non-relativistic Schrödinger equation for spinless particles in an external electromagnetic field. The corresponding stochastic mechanics was viewed as a suitable reinterpretation of Newtonian mechanics. Within his framework, however, it was not immediately clear how to approach systems which do not have a classical analogue, e.g. a spin-$\frac{1}{2}$ particle or those phenomena in which particle-like properties of radiation appear, like emission and absorption of light.

In reference [1] two of the present authors formulated a rather general scheme to associate stochastic processes with time evolution of quantum observables. The basic idea of the method consists in reinterpreting the quantum mechanical continuity equations as forward Kolmogorov equations, admitting in this way a classical stochastic interpretation.

In [1] a complete discussion of the non-relativistic Pauli equation for a spin-$\frac{1}{2}$ particle was successfully given. The same point of view proved to be very effective in the stochastic description of absorption and emission phenomena given in [2]. The above scheme, therefore, appears as a powerful heuristic principle which can be used to explore whether all aspects covered by quantum mechanics admit also a classical probabilistic interpretation.

The next obvious step is the study of a relativistic wave equation. As Dirac did in the 1920s we discard the Klein-Gordon equation as it does not possess a positive density. On the other hand the Dirac equation does have a positive probability density associated with a conserved current and therefore lends itself more easily to a stochastic interpretation.

In this paper we construct the stochastic mechanics for a Dirac particle interacting with an arbitrary external electromagnetic field. To every nowhere vanishing solution of the Dirac equation in two spacetime dimensions we associate unambiguously a stochastic process with continuous trajectories but discontinuous velocity which provides a truly stochastic interpretation of the zitterbewegung.

The problem of a path integral description of the Dirac equation [3] has recently attracted considerable attention from several people [4-7]; in particular, Gaveau *et al* [6] studied, by probabilistic methods, a 'heat' equation formally related to the two-dimensional free Dirac theory. Our aim here is different as we give, in the spirit of Nelson's stochastic mechanics [8, 9], a stochastic interpretation of the true Dirac equation in real time in two-dimensional Minkowski space.

We construct also certain non-linear field equations whose solutions ('drifts') determine the dynamics of the stochastic processes associated with Dirac wavefunctions.

In our construction these equations, fully equivalent to the Dirac theory, are relativistically covariant and contain only gauge invariant quantities like the electromagnetic tensor $F_{\mu\nu}$. In this connection there may be some contact with the work of des Cloizeaux [10] although his equations are different from ours and are not immediately related to a probabilistic interpretation.

2. The Dirac continuity equation as a Kolmogorov forward equation

The Dirac equation in two spacetime dimensions reads

$$\frac{\partial\psi}{\partial t}+c\alpha\frac{\partial\psi}{\partial x}+\frac{\mathrm{i}Mc^2}{\hbar}\beta\psi+\frac{\mathrm{i}e}{\hbar}(A_0+A_1\alpha)\psi=0 \tag{1}$$

where $\psi(t,x)=\binom{\psi_1}{\psi_{-1}}$. We use the Weyl representation where the matrices α and β are given by

$$\alpha=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Of course c is the speed of light, M the mass of the particle and A_0, A_1 are the covariant components of the electromagnetic potential. Using the same notation as [1] we consider the wavefunction ψ as a complex-valued function $\psi(t,x,\sigma)$ of the spacetime coordinates (t,x) and of a dichotomic variable $\sigma=\pm1$. In this language the continuity equation for the density $\rho(t,x,\sigma)=|\psi(t,x,\sigma)|^2$ is

$$\frac{\partial}{\partial t}\rho(t,x,\sigma)+c\sigma\frac{\partial}{\partial x}\rho(t,x,\sigma)+\frac{2Mc^2}{\hbar}\operatorname{Im}\{\bar\psi(t,x,-\sigma)\psi(t,x,\sigma\}=0 \tag{2}$$

where $\bar\psi(t,x,\sigma)$ is the complex conjugate of $\psi(t,x,\sigma)$ and not the spinor conjugate to $\psi=\binom{\psi_1}{\psi_{-1}}$.

Now our scheme consists in rewriting (2) as a forward Kolmogorov equation of the form

$$\frac{\partial}{\partial t}\rho(t,x,\sigma)+\sigma c\frac{\partial}{\partial x}\rho(t,x,\sigma)+r(t,x,\sigma)\rho(t,x,\sigma)-r(t,x,-\sigma)\rho(t,x,-\sigma)=0 \tag{3}$$

with $r(t,x,\sigma)\geqslant 0$.

The Kolmogorov equation (3) looks like a generalised version of

$$\frac{\partial \rho_\pm}{\partial t} \pm c\frac{\partial \rho_\pm}{\partial x} + a(\rho_\pm - \rho_\mp) = 0 \tag{4}$$

where a is a positive constant. This equation is described in [6] and it is connected to the so-called telegrapher's equation [11]. The generalisation (3) has the same stochastic interpretation as it describes the random motion of a particle moving on a line with speed c constant in magnitude and which inverts its direction of motion (zitterbewegung) at random times not necessarily Poisson distributed. Of course $r(t, x, \pm)$ represents the probability per unit time of inverting the motion at the spacetime point (t, x) when the velocity is $\pm c$.

Comparison of (2) and (3) gives an equation for the unknown functions $r(t, x, \pm)$ which must be solved under the constraint $r(t, x, \pm) \geqslant 0$. Following the same line of reasoning as [1], we find

$$r(t, x, \sigma) = \frac{Mc^2}{\hbar}\left(\left|\frac{\psi(t, x, -\sigma)}{\psi(t, x, \sigma)}\right| - \mathrm{Im}\,\frac{\psi(t, x, -\sigma)}{\psi(t, x, \sigma)}\right). \tag{5}$$

By this procedure with every nowhere vanishing† and normalised solution $\psi(t, x, \sigma)$ of the Dirac equation we associate a stochastic process $t \mapsto \bar{\xi}_t$ on the line such that

$$\mathrm{Prob}\left(\bar{\xi}_t \in B \text{ and } \frac{\mathrm{d}\bar{\xi}_t}{\mathrm{d}t} = \pm c\right) = \int_B |\psi(t, x, \pm)|^2 \,\mathrm{d}x \tag{6}$$

for every region $B \subseteq \mathbb{R}$ and at every time t, namely the process $\bar{\xi}_t$ reproduces at every time t the quantum joint probability for the space position and the speed of the particle.

The stochastic process $\bar{\xi}_t$ is given by

$$\bar{\xi}_t = \xi_0 + c\sigma_0 \int_0^t (-1)^{\bar{N}_\tau} \,\mathrm{d}\tau \tag{7}$$

where ξ_0 and σ_0 are real random variables distributed according to the density $\rho(0, x, \sigma) = |\psi(0, x, \sigma)|^2$ at the initial time $t = 0$. In the previous expression $\bar{N}_t$ is a point process counting ($\bar{N}_0 = 0$) the jumps of the velocity $\mathrm{d}\bar{\xi}_t/\mathrm{d}t = c\sigma_0(-1)^{\bar{N}_t}$ and which, in general (for non-constant $r(t, x, \sigma)$), is not pure Poisson.

Nevertheless the probability measure $\mathrm{d}\bar{\mathbb{P}}$ associated with $\bar{\xi}_t$ can be constructed quite explicitly from ξ_0, σ_0 and some auxiliary independent Poisson process N_t of unit parameter, i.e. $\mathbb{E}(\mathrm{d}N_t) = \mathrm{d}t$. Namely, if $\mathrm{d}\mathbb{P}$ is the probability measure corresponding to the process $\xi_t = \xi_0 + c\sigma_0 \int_0^t (-1)^{N_\tau} \,\mathrm{d}\tau$, by using the formula for the Radon-Nikodym derivative of $\bar{N}_t$ with respect to N_t as given by Kabanov *et al* in [14] (see also [15]), we obtain, for $0 \leqslant t \leqslant T$:

$$\mathrm{d}\bar{\mathbb{P}} = \left[\exp\left(\int_0^T \log r(t, \xi_t, c^{-1}\dot{\xi}_t)\,\mathrm{d}N_t + \int_0^T (1 - r(t, \xi_t, c^{-1}\dot{\xi}_t))\right)\right]\mathrm{d}\mathbb{P}. \tag{8}$$

Formula (8) gives an explicit construction of $\mathrm{d}\bar{\mathbb{P}}$ containing the jump probability per unit time $r(t, x, \sigma)$ which, in turn, depends on the wavefunction $\psi(t, x, \sigma)$ according to formula (5).

† This restriction, in the case of non-relativistic stochastic mechanics, has been removed by the recent work of Carlen, Meyer and Zheng [12, 13] (see also the work by Guerra [12a]). It would be interesting to study this problem for the case of the Pauli and Dirac equations.

3. Relativistic covariance and field equations

In the following we slightly change the notation. Let $x=(x^0, x^1)=(ct, x^1)$ be a point of two-dimensional Minkowski space. Under the Lorentz boost

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \to \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \Lambda\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

the Dirac spinor $\psi=\binom{\psi_1}{\psi_{-1}}$ undergoes the transformation

$$\psi(x,\sigma)\mapsto(\exp\tfrac{1}{2}\theta\sigma)\psi(\Lambda^{-1}x,\sigma). \tag{9}$$

As usual we write $\psi(x,\sigma)=\exp[R(x,\sigma)+\mathrm{i}S(x,\sigma)]$ with

$$R(x,\sigma)=\tfrac{1}{2}[R_0(x)-\sigma z(x)] \qquad S(x,\sigma)=\tfrac{1}{2}[S_0(x)-\sigma w(x)]. \tag{10}$$

Under the Lorentz transformation Λ we obtain, from (9) and (10)

$$\begin{aligned} &R_0(x)\to R_0(\Lambda^{-1}x) \qquad && S_0(x)\to S_0(\Lambda^{-1}x) \\ &w(x)\to w(\Lambda^{-1}x) && z(x)\to z(\Lambda^{-1}x)-\theta \end{aligned} \tag{11}$$

namely R_0, S_0 and w transform as scalar fields while $\partial_\mu z$ behaves like a covariant vector field.

Now we introduce the covariant vector fields ($g_{\mu\nu}=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$)

$$u_\mu(x)=-(\hbar/2M)\partial_\mu R_0(x) \qquad v_\mu(x)=-(\hbar/2M)\partial_\mu S_0(x)+(e/Mc)A_\mu(x) \tag{12}$$

which are gauge invariant and have the dimension of a velocity. From the definition it follows that $\partial_\mu u_\nu-\partial_\nu u_\mu=0$ and $\partial_\mu v_\nu-\partial_\nu v_\mu=(e/Mc)F_{\mu\nu}$ where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic tensor.

The knowledge of u_μ and v_μ is not sufficient to reconstruct the spinor ψ so we consider also the additional fields $z(x)$ and $w(x)$ expressed by

$$z(x)=\sigma[R(x,-\sigma)-R(x,\sigma)]=-\partial_\sigma R \qquad w(x)=\sigma[S(x,-\sigma)-S(x,\sigma)]=-\partial_\sigma S \tag{13}$$

where $\partial_\sigma f=\sigma[f(\sigma)-f(-\sigma)]$.

The full reconstruction, up to a phase factor and a normalisation constant, of the Dirac wavefunction $\psi(x,\sigma)$ is accomplished, in some chosen gauge A_μ, by the formula

$$\psi(x,\sigma)=\mathrm{e}^{\mathrm{i}\varphi}\sqrt{K}\exp-\left\{\frac{M}{\hbar}\left[\int_0^x u_\mu\,\mathrm{d}x^\mu+\mathrm{i}\int_0^x\left(v_\mu-\frac{e}{c}A_\mu\right)\mathrm{d}x^\mu\right]+\frac{\sigma}{2}\Big(z(x)+\mathrm{i}w(x)\Big)\right\} \tag{14}$$

where K is a normalisation constant and $\int_0^x\cdot_\mu\,\mathrm{d}x^\mu$ extends to an arbitrary oriented path in Minkowski space having 0 and x as end points.

Now the original Dirac theory is fully equivalent to the following set of non-linear field equations

$$\begin{aligned} &\partial_\mu u_\nu-\partial_\nu u_\mu=0 \\ &\partial_\mu v_\nu-\partial_\nu v_\mu=(e/Mc)F_{\mu\nu} \\ &\partial_0 z\pm\partial_1 z\pm(2M/\hbar)u_0+(2M/\hbar)u_1=-(2Mc/\hbar)(\sin w)\exp\pm z \\ &\partial_0 w\pm\partial_1 w\pm(2M/\hbar)v_0+(2M/\hbar)v_1=\pm(2Mc/\hbar)(\cos w)\exp\pm z \end{aligned} \tag{15}$$

which are gauge invariant and covariant under Lorentz transformations as $\exp\pm z$ transforms like $(\partial_0\pm\partial_1)z(\cdot)$ and $(\partial_0\pm\partial_1)w(\cdot)$.

We can now forget the Dirac equation and assume the set (15) as the starting point of our theory. With every reasonable solution of the field equations (15) we associate a stochastic process ξ_t on the line whose density $\rho(x, \pm)$ and transition probability per unit time $r(x, \pm)$ are given by

$$\rho(x, \pm) = K \exp -\left(\frac{2M}{\hbar}\int_0^x u_\mu \, dx^\mu \pm z(x)\right)$$
$$r(x, \pm) = (Mc^2/\hbar)(1 \mp \sin w(x)) \exp \pm z(x) \tag{16}$$

where the constant K is fixed by requiring that $1 = \int_{\mathbf{R}} [\rho(x, +) + \rho(x, -)] \, dx'$.

From the definitions (16) and the field equations (15) it follows that $\rho(x, \pm)$ obeys the Kolmogorov forward equation:

$$\partial_0 \rho(x, \sigma) \pm \partial_1 \rho(x, \sigma) - c^{-1} r(x, -\sigma)\rho(x, -\sigma) - c^{-1} r(x, \sigma)\rho(x, \sigma) = 0 \tag{17}$$

whose relativistic invariance is assured by the transformations

$$\rho(x, \sigma) \to e^{\sigma\theta}\rho(\Lambda^{-1}x, \sigma) \qquad r(x, \sigma) \to e^{-\sigma\theta} r(\Lambda^{-1}x, \sigma) \tag{18}$$

under the boost

$$\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}.$$

As a final remark we observe that, in a box of length L, normalised plane waves of positive (u^+) and negative (u^-) frequency are given by

$$u^+(x, \sigma) = \left(\frac{p^0 + \sigma p^1}{2p^0 L}\right)^{1/2} \exp(-\mathrm{i} p_\mu x^\mu)$$
$$u^-(x, \sigma) = \sigma\left(\frac{p^0 + \sigma \pi^1}{2p^0 L}\right)^{1/2} \exp(\mathrm{i} p_\mu x^\mu) \tag{19}$$

where

$$p_\mu p^\mu = m^2 = (Mc/\hbar)^2 \qquad \text{and} \qquad p^0 \geq Mc/\hbar > 0.$$

For such waves we get, in the positive frequency case:

$$u_\mu = 0 \qquad v_\mu = (\hbar/M)p_\mu$$
$$z = \log\left(\frac{p_0 - p^1}{p^0 + p^1}\right)^{1/2} \qquad w = 0 (\mathrm{mod}\, 2\pi) \tag{20}$$

while, for negative frequency, we obtain

$$u_\mu = 0 \qquad v_\mu = -(\hbar/M)p_\mu$$
$$z = \log\left(\frac{p^0 - p^1}{p^0 + p^1}\right)^{1/2} \qquad w = \pi (\mathrm{mod}\, 2\pi). \tag{21}$$

This result conveys some feeling about the interpretation of the fields u_μ, v_μ, z and w, namely that one can interpret u_μ and v_μ as osmotic and current velocity in the Minkowski space while w looks like an angle which changes by π when there is a jump from positive frequencies to negative ones.

This can be seen as follows in the general case (for superposition of plane waves): the transformation $\psi(x, \sigma) \to \sigma\bar{\psi}(x, \sigma)$ converts positive frequency solutions of the Dirac equation into negative frequency solutions. Under this transformation

$$u_\mu \to u_\mu \qquad v_\mu \to -v_\mu \qquad z \to z \qquad w \to w + \pi \tag{22}$$

namely w makes a jump of π.

4. Stochastic mechanics and the 'heat' equation

We end this paper by making a remark on the relation between the stochastic mechanics developed here and the 'heat' equation considered in [6].

From (20), (21) and (16) it follows that the transition probability per unit time $r(t, x, \sigma)$, and so the stochastic process ξ_t, is the same for corresponding plane waves of positive and negative frequency. This means that, for stationary states, the stochastic process ξ_t does not depend on the sign of the energy.

In particular, for a particle at rest for which $p^\mu = (Mc/\hbar, 0)$, we have:

$$r(t, x, \sigma) = Mc^2/\hbar \tag{23}$$

and the corresponding time homogeneous Markov process ξ_t is given by

$$\xi_t = \xi_0 + \sigma c \int_0^t (-1)^{N_s}\, ds \tag{24}$$

where N_t is a Poisson process with parameter $a = Mc^2/\hbar$.

This process is exactly the same as considered by Gaveau *et al* [6] where it was associated with the 'heat' equation:

$$\frac{\partial u(t, x, \sigma)}{\partial t} = -\sigma \frac{\partial u(t, x, \sigma)}{\partial x} - \frac{Mc^2}{\hbar}[u(t, x, \sigma) - u(t, x, -\sigma)]. \tag{25}$$

The previous equation is obtained from the free Dirac theory by making the position $u(t, x, \sigma) = (\exp \mathrm{i}(Mc^2/\hbar)t)\psi(t, x, \sigma)$ followed by the formal analytic continuation

$$t \to -\mathrm{i}t, \qquad c \to \mathrm{i}c \tag{26}$$

and it can be solved by the probabilistic formula

$$u(t, x, \sigma) = \mathbb{E}\left(u_0\left(x + \sigma c \int_0^t (-1)^{N_s}\, ds, \sigma(-1)^{N_t} \right)\right) \tag{27}$$

where $\mathbb{E}(\cdot)$ is the expectation with respect to the Poisson process N_t and $u_0(x, \pm)$ the initial datum for (25).

In this manner one can see that the 'ground-state' process (24) solves the imaginary time free Dirac equation by analogy with the case of the non-relativistic stochastic mechanics.

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